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# Generalized scaling for models with multiple absorbing states

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**Abstract.** At a continuous transition into a non-unique absorbing state, particle systems may exhibit non-universal critical behaviour, in apparent violation of hyperscaling. We propose a generalized scaling theory for dynamic critical behaviour at a transition to an absorbing state, which is capable of describing exponents which vary according to the initial configuration. The resulting hyperscaling relation is supported by simulations of two lattice models.

Non-equilibrium phase transitions continue to elicit great interest from physical and biological scientists. In the hope of obtaining a better understanding of non-equilibrium critical phenomena, models exhibiting phase transitions to an absorbing state are under intensive study in statistical physics. The dichotomy between an absorbing (dead, inactive) state and an active one arises naturally in such diverse areas as catalysis [1], epidemiology [2–4] and the transition to turbulence [5].

The essential features of the phase transition are typified by the contact process (CP) [2]. In the CP, each site of the lattice $Z^d$ is either occupied or vacant. Occupied sites become vacant at a unit rate, whilst a vacant site $i$ becomes occupied at a rate $\lambda q_i$, with $q_i$ the fraction of occupied nearest neighbours of $i$. Evidently, the vacuum is absorbing. The growth rate $\lambda$ determines the ultimate survival of the population: for $\lambda < \lambda_c$ the vacuum is the unique steady state, but for $\lambda > \lambda_c (\simeq 3.298$ in one dimension) there is also an active stationary state characterized by a non-zero particle density $\bar{\rho} \propto (\lambda - \lambda_c)^{\beta}$. The transition at $\lambda_c$ is a non-equilibrium critical point, belonging to the universality class of directed *percolation* (DP) [4, 6, 7] and Reggeon field theory [8]. Indeed, studies of a host of models provide ample support for the conjecture that continuous transitions to a unique absorbing state generically fall in the DP class [9, 10].

The situation regarding models with multiple absorbing states is more complex. On one hand, studies of some two-dimensional catalysis models yield critical exponents different from those of the DP [11–13]. On the other, the one-dimensional pair contact process (PCP) and dimer reaction (DR) clearly fall in the DP class, as far as *static* critical behaviour is concerned [14, 15]. (In all of these models, the number of absorbing configurations grows exponentially with lattice size.) The *dynamic* critical properties of the PCP and DR

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were, surprisingly, found to be non-universal, the exponents depending upon the nature of the initial configuration [15]. While this variation is quite regular, it appears to violate a basic hyperscaling relation amongst the exponents $\delta$, $\eta$ and $z$ (defined below), suggesting a breakdown of the well established scaling theory.

This apparent breakdown has prompted us to re-examine the scaling hypothesis for models with multiple absorbing states. We arrive at a scaling theory in which additional exponents are expected to depend upon the starting configuration and in which the exponents satisfy a generalized hyperscaling relation. The latter is verified in simulations of the DR and of a new model called the *threshold transfer process* (TTP). In what follows, we define the models, present the scaling theory and report the numerical evidence supporting it.

Simulations of the TTP permit us to study non-universal critical spreading in the context of a three-state model, providing a check on the robustness of earlier findings [15]. In the TTP, each site may be vacant, or singly or doubly occupied, corresponding to $\sigma_i = 0, 1$ or 2. In each cycle of the evolution, a site $i$ is chosen at random. If $\sigma_i = 0$, then $\sigma_i \to 1$ with probability $r$; if $\sigma_i = 1$, then $\sigma_i \to 0$ with probability $1 - r$. ('0' and '1' sites are left unchanged with probabilities $1 - r$ and $r$, respectively.) In the absence of doubly occupied sites, we have a trivial dynamics in which a fraction $r$ of the sites have $\sigma_i = 1$ in the steady state. However, if $\sigma_i = 2$, the particles may move to neighbouring sites. If $\sigma_{i-1} < 2$, one particle moves to that site; independently, a particle moves from $i$ to $i + 1$ if $\sigma_{i+1} < 2$. $\sigma_i$ is diminished accordingly in this deterministic particle-conserving transfer. Survival of the doubly occupied sites hinges on the process $(1, 2, 1) \to (2, 0, 2)$ (their number decreases or remains the same in all other events) and so depends upon the parameter $r$, which controls the particle density. The set of configurations devoid of doubly occupied sites comprises an absorbing subspace of the dynamics, which can be avoided only if $r$ is sufficiently large. Thus, we identify the density of doubly occupied sites, $\rho_2$, as the order parameter of the TTP.

We note in passing that the TTP bears some resemblance to a sandpile model devised by Manna [16] and to a forest-fire model (FFM) proposed by Bak *et al* [17, 18]. Under the correspondence: $2 \leftrightarrow$ burning tree, $1 \leftrightarrow$ live tree and $0 \leftrightarrow$ ashes, the process $1, 2, 1 \to 2, 0, 2$ describes a burning tree setting its neighbours on fire and $r$ represents the rate at which new trees emerge from the ashes. However, the TTP permits doubly occupied sites to arise only *via* transfer; there is no 'lightning' process as in the FFM. A further contrast is that a rule such as $0, 2, 0 \to 1, 0, 1$ has no analogue in the FFM. Despite certain common aspects, our model is therefore very different from the FFM.

The dimer reaction (DR), introduced in [15], is a lattice model in which sites may be either vacant or (singly) occupied; particles may not occupy adjacent sites. In each step of the process, a site $i$ is selected at random. If $i$ is occupied, or is blocked by a neighbouring particle, then nothing happens. But if $i$ is *open* (i.e. $i$ and its neighbours are vacant), a new particle appears, which may remain at $i$ or react with another particle depending on the occupancy of the nearby sites:

(i) If at least one of the second neighbours, $i - 2$, and $i + 2$, is occupied, then with probability $1 - p$ there is a reaction between the new particle and its neighbour (chosen at random if there is a choice), which removes them both; with probability $p$ there is no reaction and the new particle remains.

(ii) If both second neighbours are vacant, but at least one of the third neighbours is occupied, then a reaction with a third-neighbour particle may occur with probabilities as in case (i).

(iii) If none of the second or third neighbours is occupied, the new particle remains at site $i$.

In the DR, any configuration devoid of open sites is absorbing. The order parameter is the stationary open-site fraction $\bar{\rho}_0$.

Both the TTP and the DR exhibit continuous phase transitions to an absorbing state marked by a vanishing order parameter at a critical value of $r$ or of $p$. In this work, we are concerned with *critical spreading*, that is the evolution of a critical system from a nearly-absorbing initial configuration. The exponents describing this spreading are non-universal, i.e. they depend upon the particle density in the initial state [15]. Before reporting our numerical results, we present a scaling theory for such processes.

Following Grassberger and de la Torre [19], we consider an ensemble of trials, all starting from the same initial configuration: a single seed in an otherwise absorbing configuration. (For the contact process, this means one particle in an otherwise empty lattice; for the DR, one open site; and for the TTP, one doubly occupied site.) Let $\rho(x,t)$ denote the local order-parameter density and $\Delta$ the distance from the critical point ($\Delta = \lambda - \lambda_c$ in the contact process). In the critical region the system is characterized by a correlation length $\xi_\perp \propto \Delta^{-\nu_\perp}$ and relaxation time $\tau \propto \Delta^{-\nu_\parallel}$. At the critical point the asymptotic evolution is described by power laws; for $\Delta \neq 0$, the power laws are modified by scaling functions which depend upon the dimensionless ratios $x/\xi_\perp$ and $t/\tau$. Thus the survival probability—i.e. that a trial has evaded the absorbing state, $\rho(x) \equiv 0$—is expected to follow

$$P(t) \simeq t^{-\delta}\phi(\Delta t^{1/\nu_\parallel}) \tag{1}$$

so that $P \propto t^{-\delta}$ at the critical point. The order-parameter density (averaged over all trials) is

$$\rho(x,t) \simeq t^{\eta-\mathrm{d}z/2}F(x^2/t^z, \Delta t^{1/\nu_\parallel}) \tag{2}$$

where the $x$-dependence reflects symmetry and power-law critical spreading from the seed at $x = 0$. For $\Delta = 0$, one finds, on integrating equation (2) over space, that the mean population $n(t) \propto t^\eta$, whilst the second moment implies that the mean-square spread of the population $R^2(t) = \langle x^2\rangle_t \propto t^z$. The exponents $\delta$, $\eta$ and $z$ characterize critical spreading; several relations connect them with other exponents.

Consider first the CP, for which the ultimate survival probability $P_\infty \equiv \lim_{t\to\infty} P(t) = \bar{\rho}$, the stationary particle density [19]. Existence of the limit requires $\phi(x) \propto x^{\delta\nu_\parallel}$ for large $x$ and $\bar{\rho} \propto \Delta^\beta$ then implies the scaling relation

$$\delta = \beta/\nu_\parallel. \tag{3}$$

For $\Delta < 0$ and (large) fixed $t$ we expect $\rho(x,t) \simeq \mathrm{e}^{-x/\xi}$, which implies (since $\xi \propto \Delta^{-\nu_\perp}$) that for $v < 0$, $F(u,v) \propto \exp(-\mathrm{constant}\sqrt{u}|v|^{\nu_\perp})$. In order for $\xi$ to be time independent, we must have

$$z = 2\nu_\perp/\nu_\parallel. \tag{4}$$

Finally, note that for $\Delta > 0$, the local density at any fixed $x$, in a surviving trial, must approach $\bar{\rho}$ as $t \to \infty$. Since $\rho(x,t)$ represents an average over *all* trials, we have

$$\rho(x,t) \to P_\infty\Delta^\beta \propto \Delta^{2\beta} \tag{5}$$

as $t \to \infty$, which implies that $F(0,v) \propto v^{2\beta}$ for large $v$. Existence of a stationary state then requires that the exponents satisfy the hyperscaling relation

$$4\delta + 2\eta = \mathrm{d}z. \tag{6}$$

We turn now to models such as the PCP [14], the DR [15] and the TTP, which possess a multitude of absorbing configurations. Absorbing configurations in the PCP and DR can have various particle densities; the analogous variable in the TTP is the density $\rho_1$ of singly occupied sites. We refer to this aspect of the (near-absorbing) initial configuration in a critical spreading process as the 'initial density', $\phi_i$. One value of $\phi_i$ is special in these models: the 'natural' particle density $\phi_c$ of the quasistationary critical process. (For the DR, $\phi_c \simeq 0.418$; for the TTP, $\phi_c \simeq 0.69$.) Simulations indicate that in each of these one-dimensional models, the *static* critical behaviour belongs to the directed percolation class but that the exponents $\delta$, $\eta$ and $z$ are *non-universal*, varying continuously with initial density. (The critical point $\lambda_c$, by contrast, does not change as $\phi_i$ is varied.) Only when $\phi_i = \phi_c$ do the exponents assume DP values and only then do they satisfy equation (6). Rather than interpreting this as a violation of hyperscaling, we shall argue that in these models the scaling hypothesis must be modified, leading to a generalized hyperscaling relation.

We assume that, as in models with a unique absorbing state, the order-parameter density has the scaling form

$$\rho(x,t) \simeq t^{\eta'-dz'/2} G(x^2/t^{z'}, \Delta t^{1/\nu'_\|}) \tag{7}$$

where the primed exponents are functions of $\phi_i$. Similarly, we suppose the survival probability follows

$$P(t) \simeq t^{-\delta'} \Phi(\Delta t^{1/\nu'_\|}). \tag{8}$$

Since the stationary distribution is unique, we have, as before, that

$$\rho(x,t) \to P_\infty(\phi_i)\Delta^\beta \tag{9}$$

as $t \to \infty$, with $\beta$ the usual DP exponent. However, there is no reason to suppose that $P_\infty(\phi_i) \propto \overline{\rho}$, when $\phi_i \neq \phi_c$. In fact, if this were so, we would have $\delta'\nu'_\| = \beta$, implying that the primed exponents satisfy equation (6). Since they do not, we conclude that the exponent governing the ultimate survival probability must also depend upon $\phi_i$, i.e. $P_\infty \propto \Delta^{\beta'}$ with $\beta' = \delta'\nu'_\|$. By the same arguments as those applied to the CP, we find

$$z' = 2\nu'_\perp/\nu'_\| \tag{10}$$

where we have introduced exponents $\nu'_\|$ and $\nu'_\perp$ which govern the mean lifetime and spatial extent of a cluster grown from a single seed. The asymptotic behaviour of the order-parameter density is now $\rho(x,t) \to \Delta^{\beta+\beta'}$ and $G(0,y) \propto y^{\beta+\beta'}$ for large $y$, which implies the generalized hyperscaling relation

$$2\left(1+\frac{\beta}{\beta'}\right)\delta' + 2\eta' = dz'. \tag{11}$$

We have verified equation (11) in simulations of the DR and the TTP. Using time-dependent simulations (for $t \leqslant 2000$ and samples of $10^5$ to $8 \times 10^5$ trials) we determined the critical point of the TTP as $r_c = 0.6894(3)$. Analysis of steady-state data for $\bar{\rho}_2$, as shown in figure 1, then yields $\beta = 0.279(5)$, in good agreement with the value for DP in $1+1$ dimensions, $\beta = 0.2769(2)$ [20, 21]. The exponents $\delta'$, $\eta'$ and $z'$ may be determined from simulations at $r_c$ using an initial configuration very close to the absorbing state. We studied various initial densities, including $\phi_i = 0.69$, the natural value. The ultimate survival-probability exponent $\beta'$ was determined from similar studies using $r$ at slightly above the critical value. The simulations begin with one doubly occupied site at the origin; the remaining sites are taken as occupied or vacant, independently, with probabilities $\phi_i$ and $1 - \phi_i$, respectively. The dynamics is restricted to an active region defined as follows. Let $\Lambda_i$ be the set of all sites which are doubly occupied or have a doubly occupied neighbour after the $i$th step of the trial. ($\Lambda_0$ comprises the origin and its neighbours.) The site to be updated at step $i+1$ is selected at random from $\cup_{j=0}^{i}\Lambda_j$. Thus, the evolution proceeds on an expanding set within the 'light-cone' emanating from the origin. As in the DR and the PCP, distant sites are not updated until the active region reaches their neighbourhood. Figure 2 shows a local-slope analysis for $\delta$, i.e. a plot of $\delta(t) \equiv \ln[P(mt)/P(t)]/\ln m$ against $t^{-1}$, for various initial densities. (In this study we used $m = 5$.) In figure 3 we show typical results for the ultimate survival probability, leading to an estimate for $\beta'$.

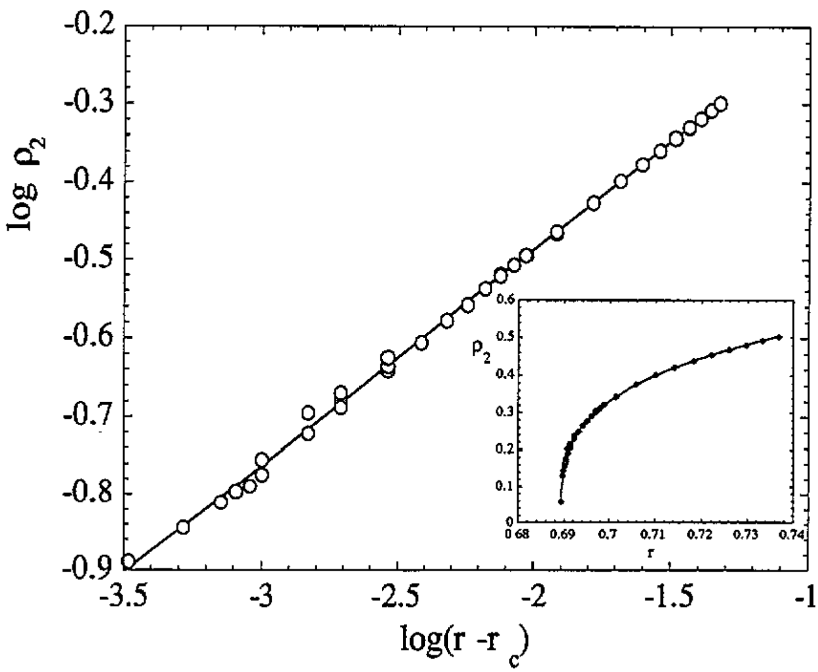


**Figure 1.** Steady-state order-parameter density, $\bar{\rho}_2$, against distance from critical point in the TTP: main diagram, log–log plot; inset, linear plot.

The simulation procedure for the DR is described in [15]. On the basis of more extensive studies of the half-life $\tau$, on lattices of up to 1000 sites, we now find $p_c = 0.264\,01(2)$, consistent with the earlier result of 0.264 00(5). We used $p$ values slightly below the critical point ($p_c - p \leqslant 0.05$) in determinations of $\beta'$ at the four initial densities studied in [15]. According to equation (8), in a plot of $\tilde{P} \equiv \Delta^{-\beta'}P$ against $\tilde{t} \equiv \Delta t^{1/\nu'_\parallel}$, data for various $\Delta$ (for a particular $\phi_i$) should fall on a single curve. Figure 4 shows a reasonably good collapse of the data for each of four initial densities.

Our results for the exponents in the TTP and DR are given in table 1 together with a test of the new hyperscaling relation (11). Evidently, it is confirmed to within the precision of the data. (By contrast, the DP hyperscaling relation, equation (6), is clearly violated.) Thus,

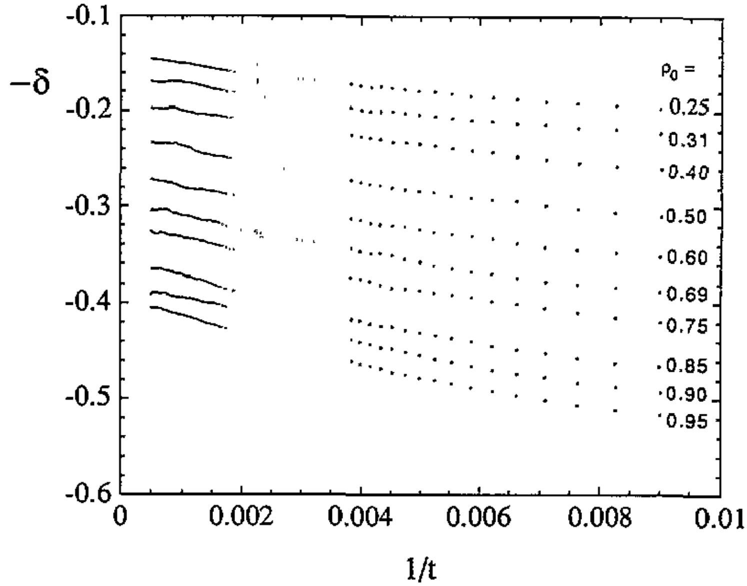


**Figure 2.** Local-slope analysis of the survival probability data for various $\phi_i$ values in the TTP. $\delta$ is estimated from the $t \to \infty$ intercept.

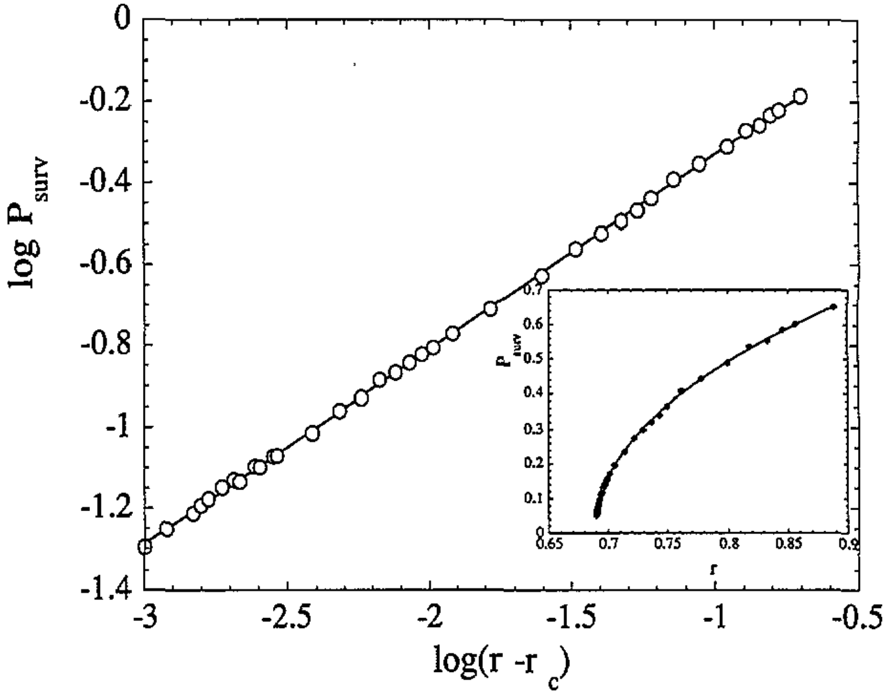


**Figure 3.** Ultimate survival probability against distance from the critical point in the TTP for initial density $\phi_i = 0.4$.

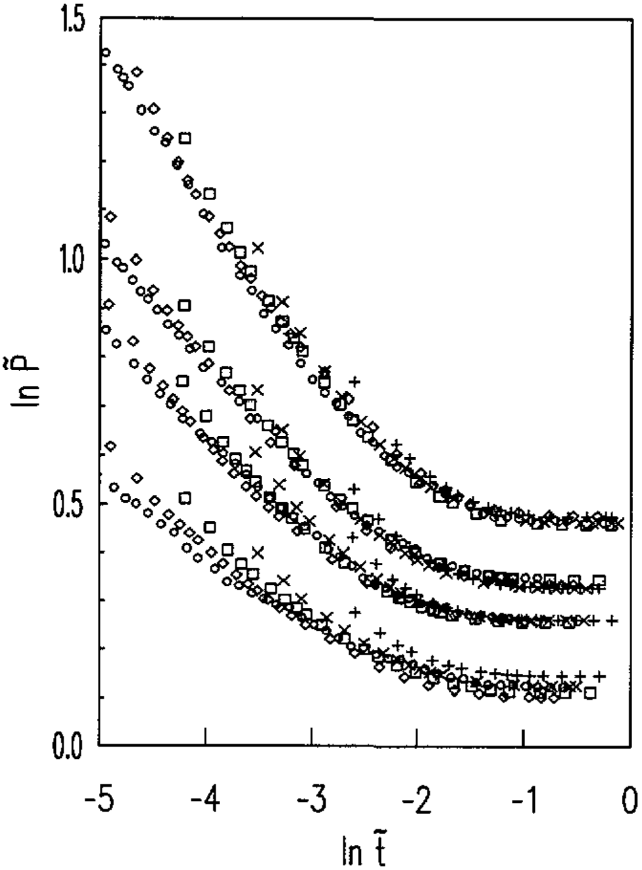


**Figure 4.** Scaling plot of the survival probability in the DR: +, $\Delta = 0.05$; ×, $\Delta = 0.02$; □, $\Delta = 0.01$; ◇, $\Delta = 0.005$; ○, $\Delta = 0.002$. Initial densities (top to bottom) $\phi_i = 0.5, 0.418, 0.38$, and 0.333.

the spreading and survival exponents for transitions to a non-unique absorbing state may be described using conventional scaling theory, properly generalized to allow for a dependence upon the initial density.

The dependence of $\delta'$, $\eta'$ and $\beta'$ upon the initial density is quite pronounced; that of $z$ much weaker. We have made no determination of $\nu'_\perp$ and our results for $\nu'_\parallel$, which come solely from the relation $\nu'_\parallel = \beta'/\delta'$, show no significant variation with $\phi_i$. (We find $\nu_\parallel = 1.80(6)$ and 1.76(6) for the TTP and the DR, respectively; the DP value is 1.74(1).) In light of equation (10), it appears that $\nu_\perp$ is not strongly dependent upon the initial density either.

Further examination of the data indicates that $\delta' + \eta'$ is also very nearly constant. This is clear from the plot of $\eta'$ against $\delta'$ for all three models (TTP, DR and PCP) shown in figure 5. (The slope of the linear best-fit is $-0.995$.) It is also worth noting that the exponents of the (two-dimensional) dimer–trimer model [12] differ from those of the DP but that $\delta + \eta$ is again the same as in the DP. (Simulations of the dimer–trimer model yield $\delta = 0.40(1)$, $\eta = 0.28(1)$, compared with 0.460(6), and 0.214(8), respectively, for two-dimensional DP [22].) Now $\delta' + \eta'$ is the exponent governing the population growth in *surviving* critical trials. Its independence of $\phi_i$ suggests that the asymptotic properties of a surviving trial are

**Table 1.** Critical exponents of the TTP and the DR. Figures in parentheses denote uncertainties in the last figure(s).

| $\phi_i$ | $\delta'$ | $\eta'$ | $z'/2$ | $\beta'$ | $\eta' + (1 + (\beta/\beta'))\delta' - z'/2$ |
|---|---|---|---|---|---|
| | | | Threshold transfer process | | |
| 0.75 | 0.136(1) | 0.347(4) | 0.632(7) | 0.250(5) | 0.00(1) |
| 0.69 | 0.161(2) | 0.319(3) | 0.632(7) | 0.279(5) | 0.00(1) |
| 0.60 | 0.192(2) | 0.288(3) | 0.630(7) | 0.356(5) | 0.00(1) |
| 0.50 | 0.227(2) | 0.246(2) | 0.623(7) | 0.426(5) | 0.00(1) |
| 0.40 | 0.270(3) | 0.204(2) | 0.622(7) | 0.497(5) | 0.00(1) |
| 0.31 | 0.299(3) | 0.169(2) | 0.617(7) | 0.556(6) | 0.00(1) |
| 0.30 | 0.303(3) | 0.170(2) | 0.621(7) | 0.567(6) | 0.00(1) |
| 0.25 | 0.316(3) | 0.161(2) | 0.624(7) | 0.591(6) | 0.00(1) |
| 0.20 | 0.342(3) | 0.133(1) | 0.622(7) | 0.640(6) | 0.00(1) |
| 0.10 | 0.371(4) | 0.097(1) | 0.615(7) | 0.705(7) | 0.00(1) |
| | | | Dimer reaction | | |
| 0.333 | 0.107(2) | 0.362(3) | 0.634(3) | 0.182(10) | 0.00(1) |
| 0.380 | 0.133(2) | 0.327(3) | 0.629(5) | 0.241(6) | −0.02(1) |
| 0.418 | 0.158(2) | 0.302(4) | 0.626(3) | 0.275(2) | −0.01(1) |
| 0.500 | 0.205(5) | 0.250(5) | 0.620(3) | 0.357(10) | −0.01(1) |

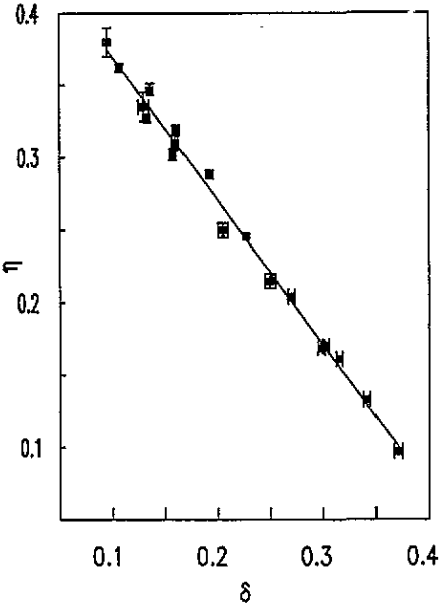


**Figure 5.** $\eta'$ against $\delta'$ for the PCP, DR and TTP. The slope of the best-fit straight line is −0.995.

not affected by the initial density. This conclusion is strongly supported by the absence of any detectable shift in the critical point as $\phi_i$ is varied. As further confirmation, one may note that, as $t \to \infty$, only a negligible fraction of a surviving cluster is actually in contact with the external density $\phi_i$. Deep inside the cluster, the particle density must approach the natural value $\phi_c$. This point of view also implies that $z'$, which describes surviving trials exclusively, should be constant. In fact, if $z'$ and $\delta' + \eta'$ are constant, then equations (10) and (11) require that $\nu'_{\parallel}$ and $\nu'_{\perp}$ are constant as well. The latter exponents will then assume their static values, which must be independent of $\phi_i$, since they describe the stationary state. We are led, by this line of argument, to a more economical description of critical spreading in which all of the exponent variation follows from a single cause: the dependence of the survival probability upon initial density. The ensuing predictions regarding exponents are consistent with our numerical results, except for a small variation in $z'$ with $\phi_i$. One may argue, however, that the results for $z'$ are affected by $\phi_i$-dependent corrections to scaling and that a more precise numerical test is needed. In summary, we believe that the most natural and parsimonious interpretation is that the initial density influences the survival probability but not the scaling properties of surviving events.

The PCP, DR and TTP all involve a second variable, $\phi$, dynamically coupled to the order parameter. A quantitative theory of the dependence survival probability, and the associated exponents $\delta'$ and $\beta'$, upon the initial density $\phi_i$ has yet to be devised. However, we can offer some intuitive basis for understanding non-universality by suggesting that, in these processes, the initial density plays a role analogous to that of a *marginal parameter*. Such parameters, invariant under renormalization group transformations, often give rise to exponents which vary continuously along a line of fixed points. In the present case, $\phi_i$ represents a property of the medium into which the process grows and which is never forgotten, since, to survive, a critical process must repeatedly invade new territory. A renormalization group transformation generally involves coarse graining (which, generally, conserves density) and rescaling. Such a transformation may be expected to leave $\phi_i$ (the density outside the active region) invariant while driving the correlation length of this region to zero. Indeed, the spreading exponents are insensitive to (short-range) correlations in the initial state [15]. A more detailed understanding of non-universality in these models may emerge when a suitable renormalization group scheme is devised.

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